Chapter 3

Structural Representations of Mechanisms

3.1 Introduction

The kinematic structure of a mechanism contains the essential information about which link is connected to which other link by what type of joint. The kinematic structure of a mechanism can be represented in several different ways. Some methods of representation are fairly straightforward, whereas others may be rather abstract and do not necessarily have a one-to-one correspondence. In this chapter various methods of representation of the kinematic structure of a mechanism or kinematic chain are described. For convenience, the following assumptions are made for all methods of representation.

1. For simplicity, all parallel redundant paths in a mechanism will be illustrated by a single path. Parallel paths are usually employed for increasing load capacity and achieving better dynamic balance of a mechanism. For example, Figure 3.1 depicts the components of a basic planetary gear train whose schematic diagram is shown in Figure 3.2a. Although the gear train has four planets, the structural representation is sketched with only one, as illustrated in Figure 3.2b. Similarly, when a link is supported by several coaxial bearings, only one will be shown.

2. All joints are assumed to be binary. A multiple joint will be substituted by a set of equivalent binary joints. In this regard, a ternary joint will be replaced by two coaxial binary joints, a quaternary joint will be replaced by three coaxial binary joints, and so on.

3. Two mechanical components rigidly connected for the ease of manufacturing or assembling will be considered and shown as one link. For example, two gears keyed together on a common shaft to form a compound gear set will be treated as one link.

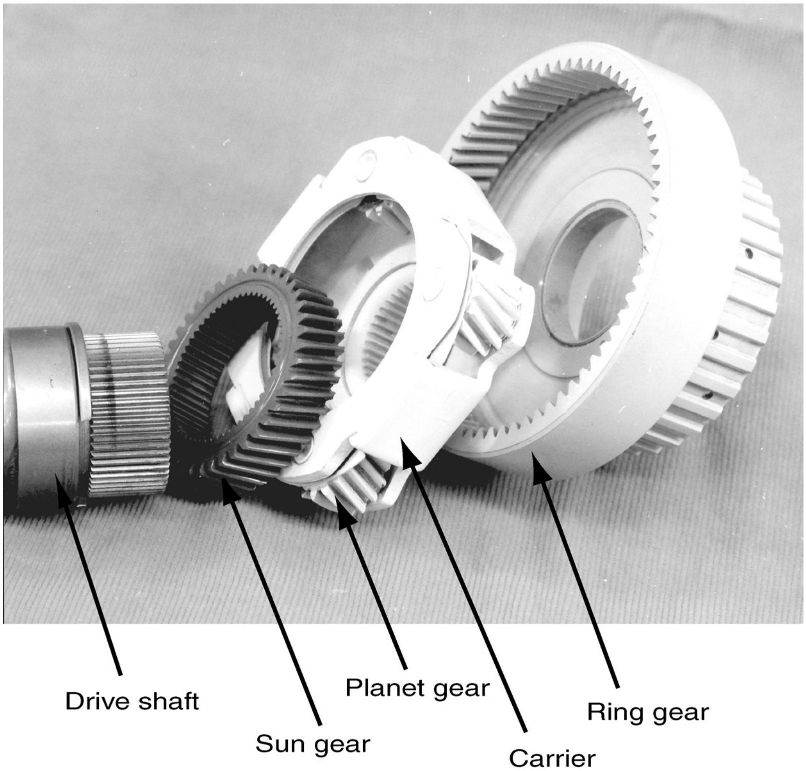


FIGURE 3.1
A basic planetary gear train.

3.2 Functional Schematic Representation

Functional schematic representation refers to the most familiar cross-sectional drawing of a mechanism. Shafts, gears, and other mechanical elements are drawn as such. For clarity and simplicity, only those functional elements that are essential to the structural topology of a mechanism are shown.

Two functional schematics representing different physical embodiments might sometimes share the same structural topology. For example, Figures 3.3a and b show the schematic diagrams of two different mechanisms. Each of the two mechanisms contains four links connected by two revolute, one spherical, and one cylindric joint. The two revolute joint axes in Figure 3.3a intersect at an oblique angle, whereas the two revolute joint axes in Figure 3.3b are perpendicular to each other with an offset distance. Both mechanisms are capable of converting the rotational motion of link 2 into the reciprocating and oscillatory motions of link 4. These two mechanisms are different in physical embodiment, but their structural topologies are identical.

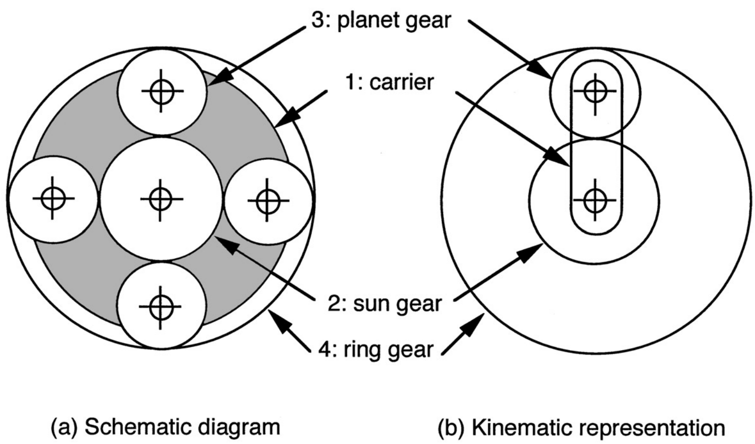


FIGURE 3.2
Schematic diagram and kinematic representation.

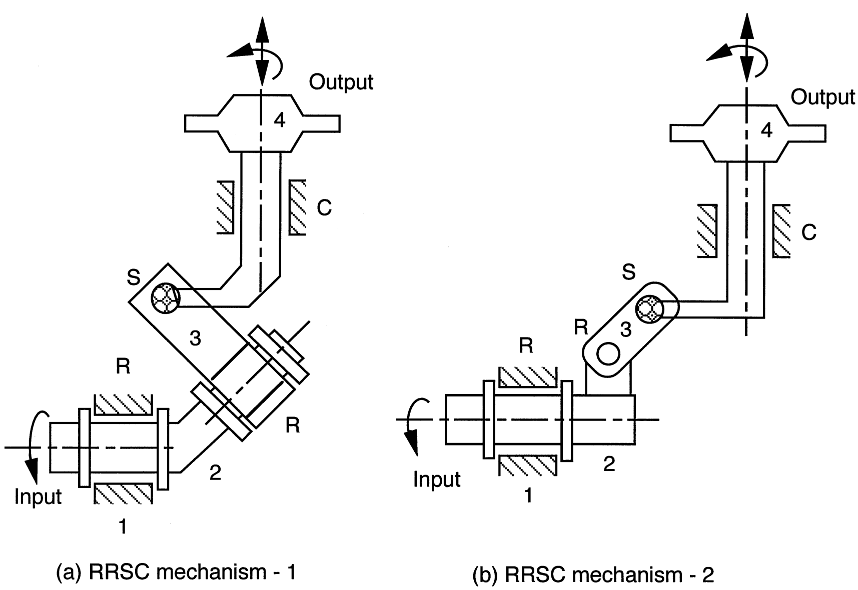


FIGURE 3.3
Functional schematics of two $RRSC$ spatial mechanisms.

Similarly, various planetary gear trains with internal versus external gear mesh may share identical structural topology. Figure 3.4a shows the functional schematic of a spur gear set with an external gear mesh, whereas Figure 3.4b shows the functional schematic of another spur gear set with an internal gear mesh. Each of these two gear sets contains three links. Gear 2 meshes with gear 3, whereas link 1 serves as the carrier. Together, they form a one-dof gear train. In the side view of a gear pair, we use two short parallel lines to indicate the gear mesh. These two gear sets are different in design. However, their structural topologies are identical, to some extent, as will be described in the following section.

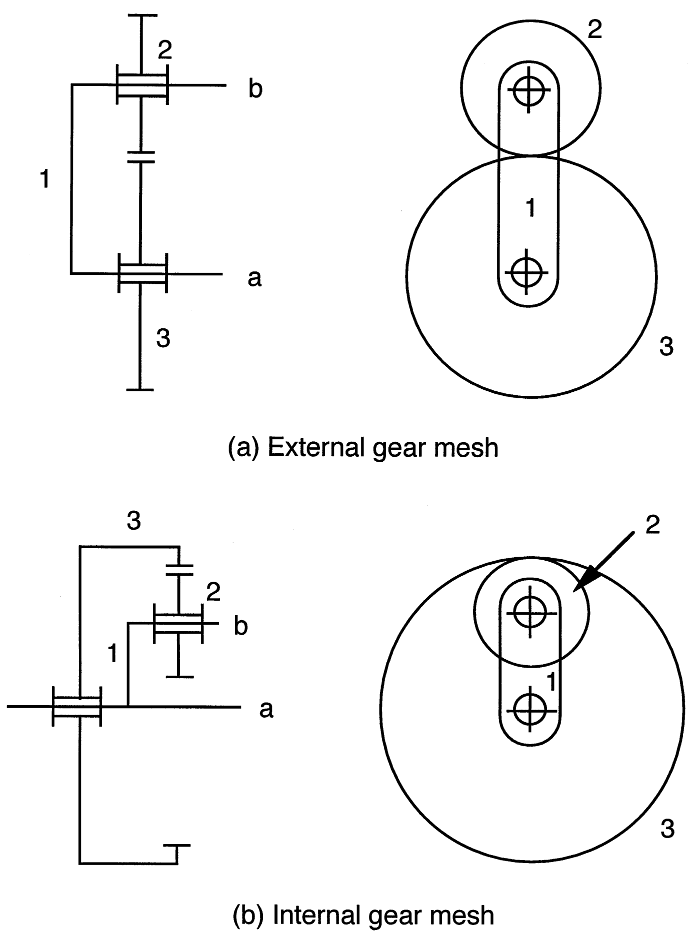


FIGURE 3.4
Functional schematics of two gear sets that share the same structural topology.

3.3 Structural Representation

In a *structural representation,* each link of a mechanism is denoted by a polygon whose vertices represent the kinematic pairs. Specifically, a binary link is represented by a line with two end vertices, a ternary link is represented by a cross-hatched triangle with three vertices, a quaternary link is represented by a cross-hatched quadrilateral with four vertices, and so on. Figure 3.5 shows the structural representation of a binary, ternary, and quaternary link. The vertices of a structural representation can be colored or labeled for the identification of pair connections. For example, *plain vertices* shown in Figure 3.5 denote revolute joints, whereas *solid vertices* denote gear pairs.

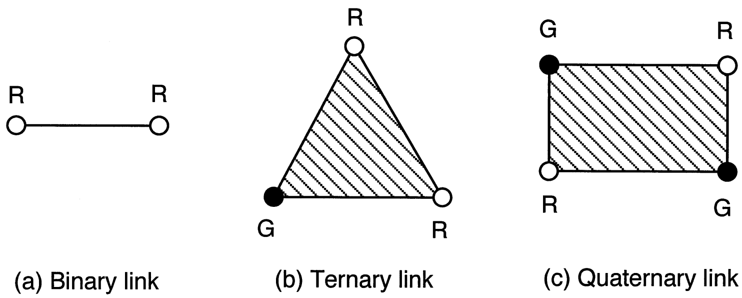


FIGURE 3.5
Structural representation of links.

The structural representation of a mechanism is defined similarly, except that the polygon denoting the fixed link is labeled accordingly. Unlike the functional schematic representation, the dimensions of a mechanism, such as the offset distance and twist angle between two adjacent links, are not shown in the structural representation.

Figure 3.6 shows the structural representation of the two $RRSC$ spatial mechanisms depicted in Figure 3.3, where the edge label denotes the link number and the vertex label denotes the joint type. Figure 3.6 shows that the four links are connected in a closed loop by revolute, revolute, spherical, and cylindric joints. We conclude that both mechanisms shown in Figure 3.3 share the same structural topology.

Figure 3.7 depicts the side view of the planetary gear train shown in Figure 3.2 and the corresponding structural representation. We note that, at this level of abstraction, the type of gear mesh is not specified. In this regard, the kinematic structure shown in Figure 3.7 may be sketched in more than one functional schematic. Either gear pair can assume either external or internal gear mesh. Hence, there is no one-to-one correspondence between the functional schematic and the structural representation. To distinguish the difference requires one additional level of abstraction. For example,

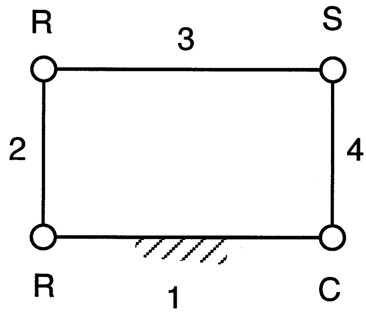


FIGURE 3.6
Structural representation of the two $RRSC$ mechanisms shown in Figure 3.3.

we may use the symbol G_i to represent an internal gear mesh and G_o an external gear mesh.

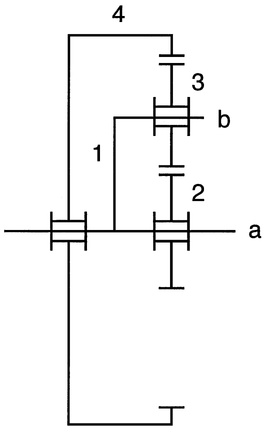
Figure 3.8 shows the schematics and structural representations of some link assortments frequently used in geared kinematic chains.

3.4 Graph Representation

Since a kinematic chain is a collection of links connected by joints, this link and joint assemblage can be represented in a more abstract form called the *graph representation.* In a graph representation, the vertices denote links and the edges denote joints of a mechanism. The edge connection between vertices corresponds to the pair connection between links. To distinguish the differences between various pair connections, the edges can be labeled or colored. For example, the gear pairs in a gear train can be represented by thick edges and the turning pairs (revolute joints) by thin edges. Furthermore, the thin edges can be labeled according to the locations of their axes.

The graph of a mechanism is defined similarly with only one addition; the vertex denoting the fixed link is labeled accordingly, usually with two small concentric circles. For example, Figure 3.9 depicts a graph representation of the $RRSC$ mechanism shown in Figure 3.3. The vertices shown in Figure 3.9 are numbered from 1 to 4 representing links 1 to 4, respectively, and the edges are labeled as R, R, S, and C according to the pair connections between links.

Similarly, Figure 3.10 illustrates the graph representation of the planetary gear set shown in Figure 3.2. In Figure 3.10, the thick edges denote gear pairs and the thin edges denote turning pairs. Since the type of gear mesh is not specified at this level of abstraction, the graph shown in Figure 3.10 can also represent a gear set with two external gear meshes or two internal meshes.



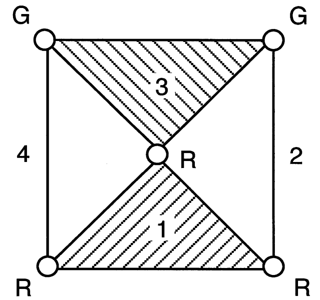


FIGURE 3.7
Functional and structural representations of the planetary gear set shown in Figure 3.2.

The sketching of a graph from a mechanism is very straightforward. However, the inverse process, that is, the sketching of a mechanism from the graph, requires some practice to achieve nice proportions. In general, a single graph can be sketched into several different mechanism embodiments.

Figure 3.11 shows three different kinematic representations of four epicyclic gear trains.

3.4.1 Advantages of Using Graph Representation

The advantages of using the graph representation are:

1. Many network properties of graphs are directly applicable. For example, we can apply Euler's equation to obtain the *loop mobility criterion* of mechanisms directly.

2. The structural topology of a mechanism can be uniquely identified. Using graph representation, the similarity and difference between two different mechanism embodiments can be easily recognized.

3. Graphs may be used as an aid for the development of computer-aided kinematic and dynamic analysis of mechanisms. For example, Freudenstein and Yang [7] applied the *theory of fundamental circuits* for the kinematic and static force analysis of planar spur gear trains. The theory was subsequently extended to the kinematic analysis of bevel-gear robotic mechanisms [12]. Recently,

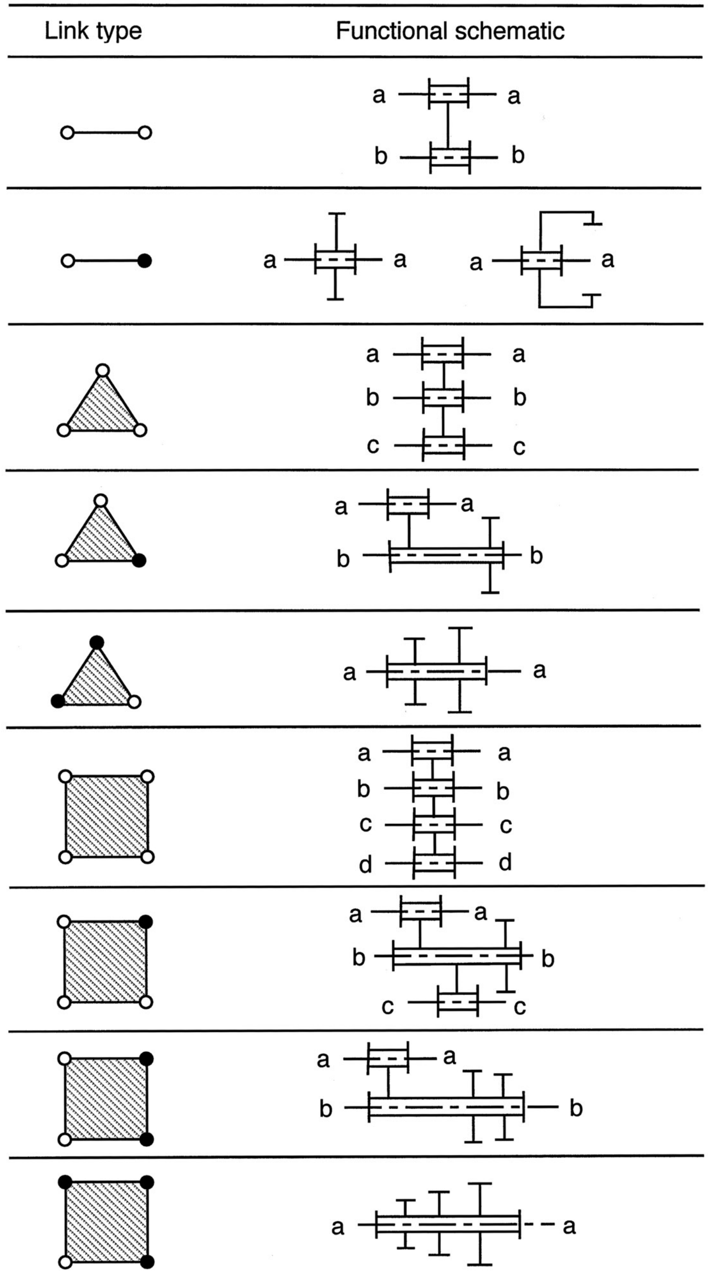


FIGURE 3.8
Link assortments frequently used in geared kinematic chains.

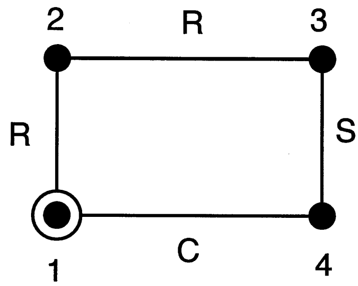


FIGURE 3.9
Graph representation of the *RRSC* mechanisms shown in Figure 3.3.

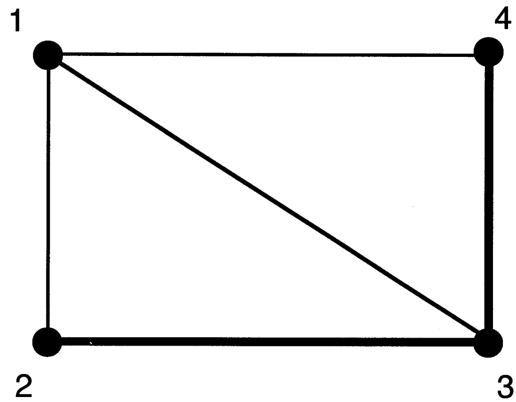


FIGURE 3.10
Graph representation of the planetary gear set shown in Figure 3.2.

a systematic methodology for the dynamic analysis of gear coupled robotic mechanisms was developed [13].

4. Graph theory may be employed for systematic enumeration of mechanisms. [1, 2, 4, 6, 8, 10, 11, 14].

5. Graphs can be used for systematic classification of mechanisms. A single atlas of graphs can be used to enumerate an enormous number of mechanisms [5, 6, 9]. This obviates the need for an individual atlas of kinematic chains tailored for each application.

6. Graphs can be used as an aid in automated sketching of mechanisms [3].

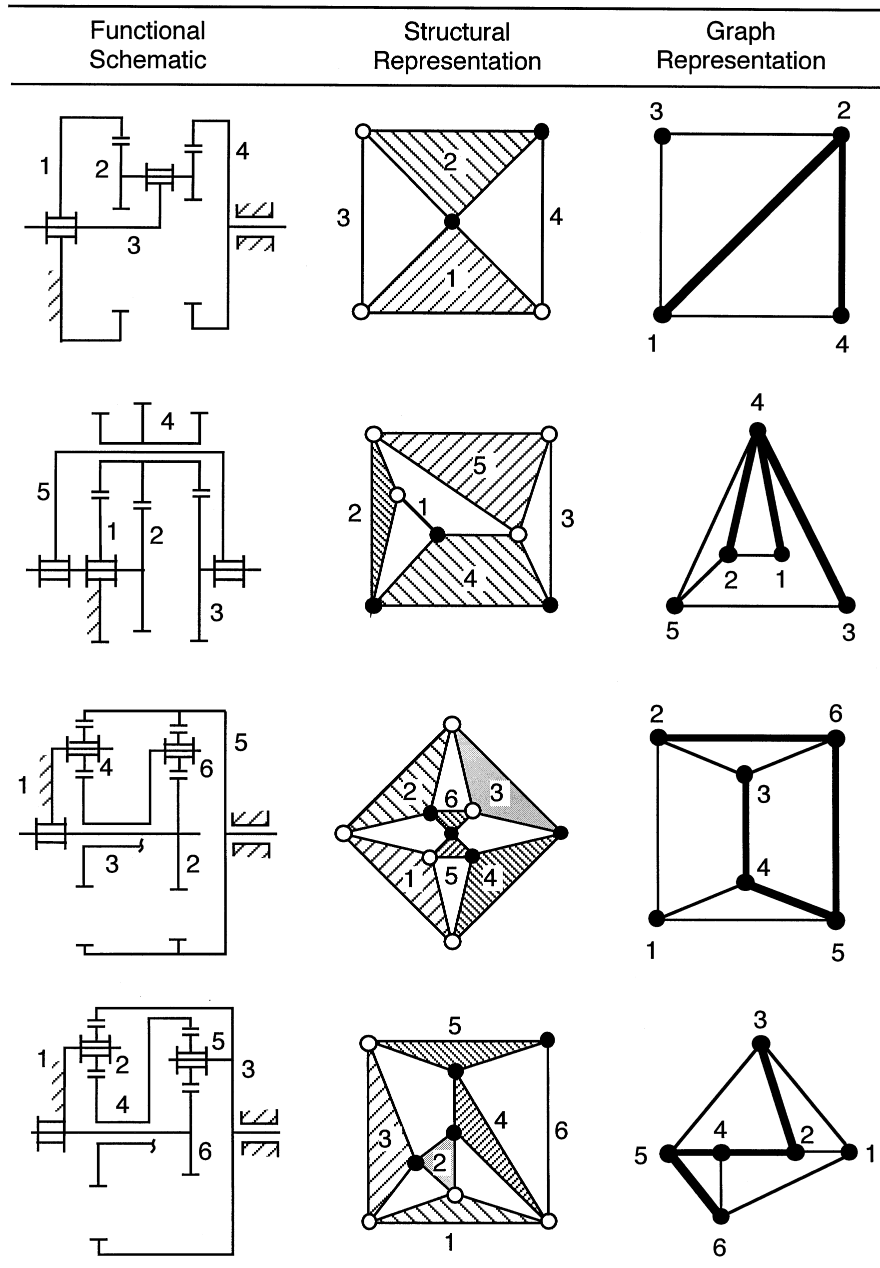


FIGURE 3.11
Kinematic representation of four epicyclic gear trains.

3.5 Matrix Representation

For convenience of computer programming, the kinematic structure of a kinematic chain is represented by a graph and the graph is expressed in matrix form. There are several methods of matrix representation as described in Chapter 2. Perhaps, the most frequently used method is the link-to-link form of adjacency matrix. Other methods of representation, such as the incidence matrix, circuit matrix, and path matrix, are also useful for the identification and classification of mechanisms. Matrix representations are particularly useful for computer aided enumeration of kinematic structures of mechanisms. In the following, we briefly describe the adjacency and incidence matrix representations of kinematic chains.

3.5.1 Adjacency Matrix

The links of a kinematic chain are numbered sequentially from 1 to n. Since in the graph, representation vertices correspond to links and edges correspond to joints, the link-to-link *adjacency matrix,* A, is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if link } i \text{ is connected to link } j \text{ by a joint}\,, \\ 0 & \text{otherwise (including } i = j)\,. \end{cases} \tag{3.1}$$

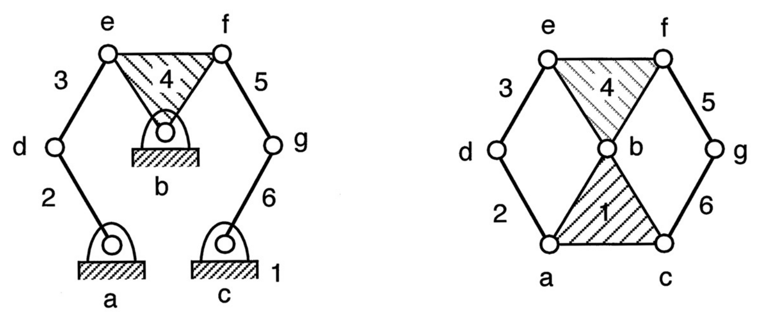
By definition, the adjacency matrix is an $n \times n$ symmetric matrix with zero diagonal elements. The matrix determines the structural topology of a kinematic chain up to structural isomorphism. For example, the link-to-link adjacency matrix of the spur-gear set shown in Figure 3.2 is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \tag{3.2}$$

The matrix representation given by Equation (3.2) provides no distinction for the types of joint used in a mechanism. The (2, 3) element in Equation (3.2) simply provides the information that link 2 is connected to link 3 by a joint. It does not give information about the type of joint. To resolve this problem, one additional level of abstraction is needed. We can employ different numerals and/or letters to denote the joint types. For example, we may use the numeral "1" to represent a turning pair and the letter "g" to denote a gear pair. Using this notation, the adjacent matrix of the planetary gear set shown in Figure 3.2 becomes

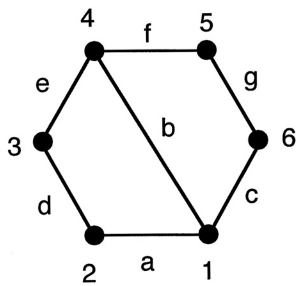
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & g & 0 \\ 1 & g & 0 & g \\ 1 & 0 & g & 0 \end{bmatrix}. \tag{3.3}$$

As a second example, Figure 3.12 shows the functional schematic, kinematic structure, graph, and adjacency matrix representations of a Watt linkage.



(a) Functional schematic

(b) Structural representation



(c) Graph representation

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(d) Adjacency matrix

FIGURE 3.12
Watt mechanism and its kinematic representations.

3.5.2 Incidence Matrix

Another useful matrix representation is the *incidence matrix, B*. In addition to labeling the links, the joints are labeled as well. In an incidence matrix each row represents a link, whereas each column denotes a joint as outlined below.

$$B = \overset{\text{joint j}}{\begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,1} & b_{n,2} & \cdots & b_{n,m} \end{bmatrix}} \text{ link i}$$

where

$$b_{ij} = \begin{cases} 1 & \text{if link } i \text{ contains joint } j \text{ ,} \\ 0 & \text{otherwise .} \end{cases} \tag{3.4}$$

The incidence matrix also determines the structural topology of a kinematic chain up to structural isomorphism. Figure 3.13 shows the functional schematic and graph

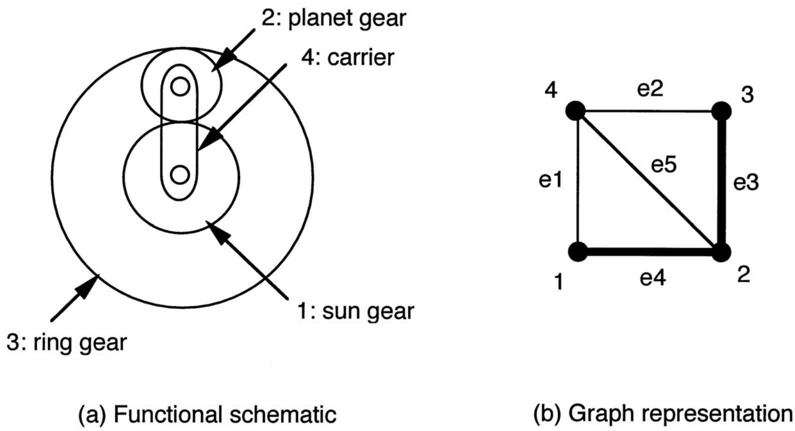


FIGURE 3.13
A planetary gear train and its graph representation.

representation of a planetary gear train with its edges labeled from $e1$ to $e5$. The incidence matrix is given by

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}. \tag{3.5}$$

3.6 Summary

A kinematic chain is an assemblage of links connected by joints. The study of the nature of connection among various links of a kinematic chain is called the *structural analysis* or *topological analysis*. To facilitate the analysis, several methods of representation of the kinematic structure were described. The study includes the functional schematic representation, structural representation, graph representation, and various matrix representations.

References

[1] Buchsbaum, F. and Freudenstein, F., 1970, Synthesis of Kinematic Structure of Geared Kinematic Chains and other Mechanisms, *Journal of Mechanisms,* 5, 357–392.

[2] Chatterjee, G. and Tsai, L.W., 1994, Enumeration of Epicyclic-Type Automatic Transmission Gear Trains, *SAE 1994 Trans., Journal of Passenger Cars,* Sec. 6, 103, 1415–1426.

[3] Chatterjee, G. and Tsai, L.W., 1996, Computer Aided Sketching of Epicyclic-Type Automatic Transmission Gear Trains, *ASME Journal of Mechanical Design,* 118, 3, 405–411.

[4] Erdman, A.G. and Bowen, J., 1981, Type and Dimensional Synthesis of Casement Window Mechanism, *ASME Mechanical Engineering,* 103, 46–55.

[5] Fang, W.E. and Freudenstein, F., 1988, The Stratified Representation of Mechanisms, in *Proceedings of the ASME Mechanisms Conference: Trends and Developments in Mechanisms, Machines, and Robotics,* Cambridge, MA, 1, 115–124.

[6] Freudenstein, F. and Maki, E.R., 1979, Creation of Mechanisms According to Kinematic Structure and Function, *Journal of Environmental and Planning B,* 6, 375–391.

[7] Freudenstein, F. and Yang, A.T., 1972, Kinematics and Statics of a Coupled Epicyclic Spur-Gear Train, *Mechanisms and Machine Theory,* 7, 263–275.

[8] Lin, C.C. and Tsai, L.W., 1989, The Development of an Atlas of Bevel-Gear Type Spherical Wrist Mechanisms, in *Proceedings of the First National Conference on Applied Mechanisms and Robotics,* Cincinnati, OH, Paper No. 89-AMR-2A-3.

[9] Mayourian, M. and Freudenstein, F., 1984, The Development of an Atlas of the Kinematic Structures of Mechanisms, in *Proceedings of the ASME Mechanisms Conference,* Cambridge, MA, Paper No. 84-DET-21.

[10] Sohn, W., 1987, A Computer-Aided Approach to the Creative Design of Mechanisms, Ph.D. Dissertation, Dept. of Mechanical Engineering, Columbia University, New York, NY.

[11] Tsai, L.W., 1987, An Application of the Linkage Characteristic Polynomial to the Topological Synthesis of Epicyclic Gear Trains, *ASME Journal of Mechanisms, Transmissions, and Automation in Design,* 109, 3, 329–336.

[12] Tsai, L.W., 1988, The Kinematics of Spatial Robotic Bevel-Gear Trains, *IEEE Journal of Robotics and Automation,* 4, 2, 150–156.

[13] Tsai, L.W., Chen, D.Z., and Lin, T.W., 1998, Dynamic Analysis of Geared Robotic Mechanisms Using Graph Theory, *ASME Journal of Mechanical Design,* 120, 2, 240–244.

[14] Yan, H.S. and Chen, J.J., 1985, Creative Design of a Wheel Damping Mechanism, *Mechanism and Machine Theory,* 20, 6, 597–600.

Exercises

3.1 Figure 3.14 shows a Humpage reduction gear train. Sketch the kinematic structure and corresponding graph, and derive the adjacency matrix.

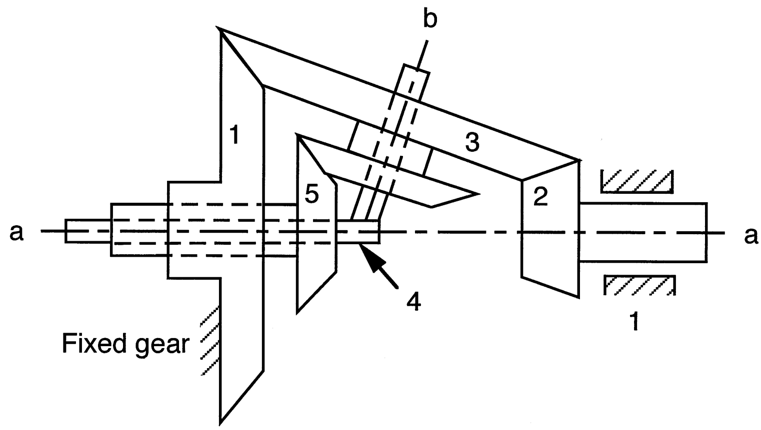


FIGURE 3.14
Humpage reduction gear.

3.2 Sketch the kinematic structure and corresponding graph, and derive the adjacency matrix for the wobble-plate mechanism shown in Figure 3.15.

3.3 Figure 3.16 shows a z-crank mechanism. Sketch the kinematic structure and corresponding graph, and then derive the incidence matrix.

3.4 Sketch the kinematic structure and corresponding graph, and derive the incidence matrix for the mechanism shown in Figure 3.17.

3.5 Figure 3.18 shows a $3RPS$ parallel manipulator. Sketch the kinematic structure and corresponding graph, and derive the adjacency matrix.

3.6 Sketch the kinematic structure and the corresponding graph, and derive the adjacency matrix for the spur gear train shown in Figure 3.19.

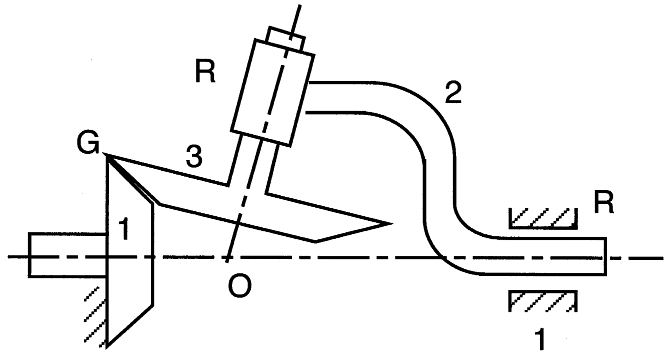


FIGURE 3.15
Wobble-plate mechanism.

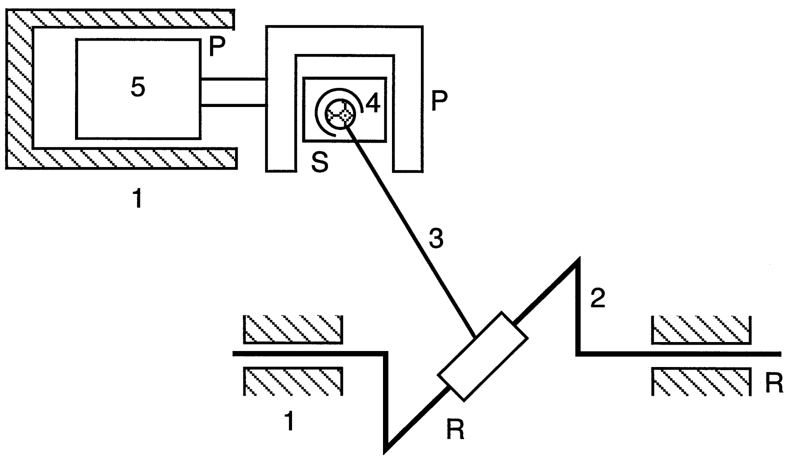


FIGURE 3.16
Z-crank mechanism.

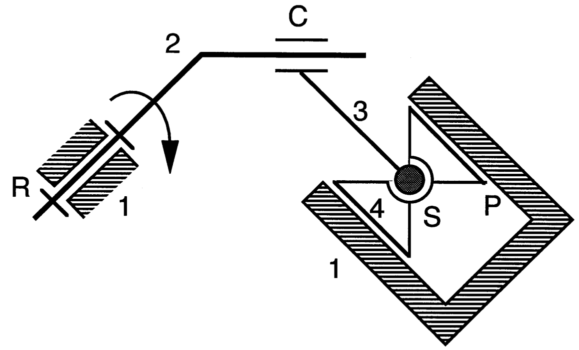


FIGURE 3.17
Spatial *RCSP* mechanism.

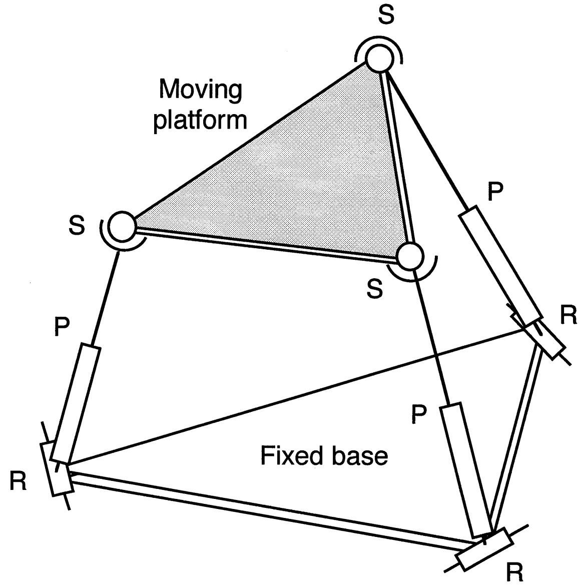


FIGURE 3.18
A 3 *RPS* parallel manipulator.

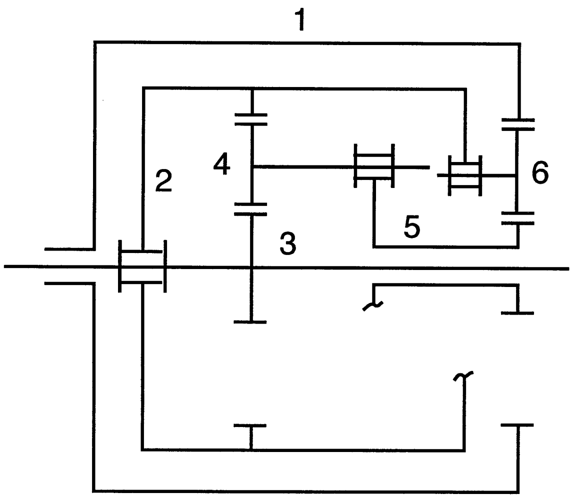


FIGURE 3.19
Six-link spur gear train.